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ANALYTICAL AND EXPERIMENTAL INVESTIGATIONS OF AN AUTOPARAMETRIC BEAM STRUCTURE

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Abstract. *This paper concerns theoretical and experimental investigations of vibrations of an autoparametric system composed of two beams with rectangular cross sections and essentially different flexibilities in two orthogonal directions. Differential equations of motion and associated boundary conditions based on the Hamilton principle of least action, are derived to third order approximation. Experimental tests of the system response, under random and harmonic excitations for the 1:4 internal resonance condition are performed, and then the most important vibration modes are extracted. It is shown that certain modes in the stiff and flexible directions of both beams may interact, and, intuitively unexpected out-of-plane motion may also appear.*

1 INTRODUCTION

Beam structures are common in mechanical and civil engineering[1,2]. Linear and nonlinear models of a single beam are studied extensively in many papers. Large vibrations of non-planar motion of inextensional beams are considered in[3]. Equation of motions with nonlinear curvatures and nonlinear inertia terms are derived systematically to third order approximation, taking into account bending about two principal axes and torsion of the beam. Reduction of the model into two differential equations is carried out by expressing twisting of the beam versus its bending in two directions. The response of such a nonlinear model when excited harmonically by an external, distributed, force is presented in[4]. Paper[5] presents the influence of parametric excitation on a single vertical beam response generated in a perpendicular plane to that of the excitation. It is shown that a few resonances can be excited simultaneously and that a weaker type of coupling can modify that of stronger coupling to a significant extent. Non-planar motion of a metal cantilever beam exited by vertical harmonic motion of the support is also presented in[6]. Bifurcation analysis shows different possible vibrations of the beam and five branches of the dynamic solution. Periodic, quasi-periodic and chaotic motions are found near the main parametric resonance. Because of a well separated torsional frequency, the influence of torsional inertia is neglected in the model. A study of nonlinear vibrations of metallic cantilever beams subjected to transverse harmonic excitations is given in[7]. Experimental and theoretical results are presented. The transfer of energy between widely spaced modes via modulation, both in the presence and absence of a one-to-one internal resonance is shown. Reduced-order models using the Galerkin discretisation are also developed to predict experimentally observed motions.

More complicated situation may appear when instead of a single beam, a set of coupled beams is to be analysed. Due to internal coupling caused by nonlinear terms resulting from nonlinear geometry and inertia, autoparametric vibrations may appear[8]. In such a case, one subsystem becomes a source of excitation to the other, and under some conditions this causes an increase of a vibration amplitude and, moreover an energy transfer between different vibration modes may take place[9]. This kind of coupling appears in, so called, “L” shaped beam structures. In-plane motion analysis of such coupled beams is presented in[10]. Derivation of the equations of motion and dynamical boundary condition are shown there for a structure flexible in one plane and stiff in the orthogonal direction. Analytical solutions are found when the strongest coupling takes place, i.e. in the neighbourhood of the principal parametric resonance and for a 2:1 internal resonance. Primary resonance of the first and the second mode, and prediction of the Hopf bifurcation, are determined analytically. Experimental tests of nonlinear motion in a coupled beam structure with quadratic nonlinearities are discussed in[11] and[12]. Periodic, quasi-periodic, and chaotic responses, predicted by theory have been confirmed. It has been shown that under a 2:1 internal resonance a very small excitation can lead to chaotic response of the structure.

Another type of “L” shaped metal beam structure is explored in[13-18]. The difference between this and the models just summarised is that the beams are coupled in such a way that their stiffnesses are essentially different in two orthogonal directions (see Fig.1). The effect of non-linear coupling between bending modes of vibration is investigated theoretically and experimentally in[13,14]. The non-linear forced vibration responses show jumps at entry and exits frequencies. Small non-linear interactions have significant effect under the 2:1 internal resonance condition. Four mode interaction exhibits large amplitudes of indirectly excited modes and saturation of the directly excited mode. Planar and non-planar motions of the vertical beam for two simultaneous internal resonance conditions are presented in[15]. The combination and internal resonances give complicated responses and intermodal energy exchange effects for small changes in external and internal tuning. Differential equations of motion have been derived taking into account bending of the horizontal beam, and bending/ torsion of the vertical beam. Paper[16] shows that violent non-synchronous torsion and bending vibrations occur as a result of the existence of quadratic non-linear coupling terms and internal resonance effects caused by strong four-mode interactions.

In spite of extensive investigations of the "L" shaped beam structure, there are no literature analyses, to the authors' knowledge, that take interaction between torsion and bending in both of the coupled beams into account. The development of the mathematical modelling is particularly important if the structure is made of composite material. Additional interactions can be observed because of a natural closeness of the torsional and bending modes frequencies, which are usually well separated for metallic structures.

This paper gives an extension of the analysis of the coupled beam structure presented in papers[13-18]. The systematic derivation of the differential equations of motions and associated boundary conditions to third order approximation are given in the first part. Then, the results of preliminary experimental tests are presented which show the modal interactions and their influence on the structure's response phenomena.

2 MODEL OF THE STRUCTURE

The structure considered in this paper consists of two thin beams made of glass-epoxy composite with fibres oriented as follows: 0/90/45/-45/45/90/0 (Fig.1). Both beams are of rectangular cross-section and are fixed in such a way that their flexibilities are essentially different in the horizontal and vertical directions[18]. They are clamped together at point *C*, while the horizontal (primary) beam is fixed at the support *B* and can be excited by the shaker in the Y_1 direction. A lumped mass *A* attached at the top of the vertical (secondary) beam allows for tuning of the structure for the required dynamical conditions.

The deformed structure and the assumed coordinate systems are presented in Fig.2. The axes X_1, Y_1, Z_1 are assumed to be inertial with their origin at point *B*, while the set X_2, Y_2, Z_2 is attached to the centre of the cross section at point *C* and overlaps the principal axes of the beam cross section. Sets ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 are the principal axes of the beam cross section at arbitrary positions s_1 and s_2 for the primary and secondary beams, respectively.

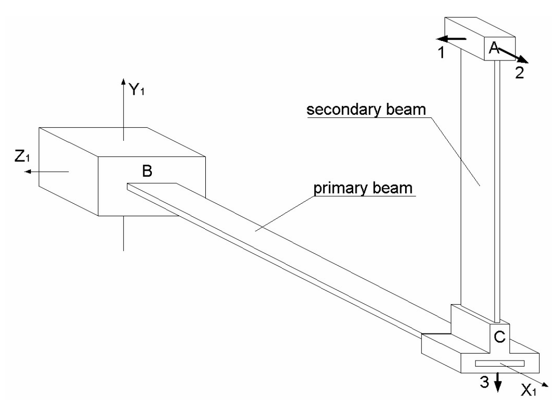

Figure 1. Model of the structure

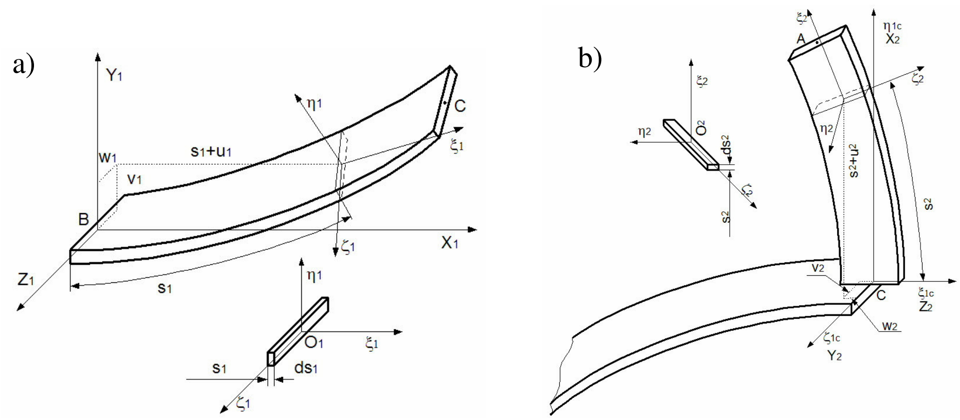

Figure 2. Deflected beam structure

The components $u_1(s_1,t)$, $v_1(s_1,t)$, $w_1(s_1,t)$ and $u_2(s_2,t)$, $v_2(s_2,t)$, $w_2(s_2,t)$ denote the elastic displacement of the cross-section centroids of the primary and secondary beams (points O_1 and O_2), while $\phi_1(s_1,t)$, $\psi_1(s_1,t)$, $\theta_1(s_1,t)$ and $\phi_2(s_2,t)$, $\psi_2(s_2,t)$, $\theta_2(s_2,t)$ represent the rotations expressed by Euler angles.

3 EQUATIONS OF MOTION

The equations of free vibration of the structure given in Fig.1 are derived by applying Hamilton's principle of least action,

$$\delta\int_{t_1}^{t_2}\left(T_1 - V_1 + F_1 + T_2 - V_2 + F_2 + T_C - V_C + T_A - V_A\right)dt = 0 \tag{1}$$

where T_1, V_1, F_1, T_2, V_2, F_2 denote the kinetic and potential energies and the constraint equations of the primary and the secondary beams, and T_C, V_C, T_A, V_A, the kinetic and potential energies of the masses C and A, respectively.

By introducing the notation $T_1 - V_1 + F_1 = \int_0^{l_1} h_1 ds_1$, $T_2 - V_2 + F_2 = \int_0^{l_2} h_2 ds_2$ equation (1) can then rewritten in the form,

$$\int_{t_1}^{t_2}\left(\int_0^{l_1}\delta h_1 ds_1 + \int_0^{l_2}\delta h_2 ds_2 + T_c - V_c + T_A - V_A\right)dt = 0 \tag{2}$$

The kinetic energy of the primary beam results from translational and rotational motions of the element shown in Fig.2a

$$T_1 = \frac{1}{2}\int_0^{l_1} \left(\rho_1 A_1 \left(V_{x_1}^2 + V_{y_1}^2 + V_{z_1}^2 \right) + I_{\xi_1} \omega_{\xi_1}^2 + I_{\eta_1} \omega_{\eta_1}^2 + I_{\zeta_1} \omega_{\zeta_1}^2 \right) ds_1 \tag{3}$$

where: ρ_1, A_1 denote density and cross-sectional area of the primary beam, $I_{\xi 1}$, $I_{\eta 1}$, $I_{\zeta 1}$ are the principal mass moments of inertia of the beam per unit length. Velocity components of the translational motion take the form,

$$V_{x_1} = \dot{u}_1 ,\ V_{y_1} = \dot{v}_1 ,\ V_{z_1} = \dot{w}_1 \tag{4}$$

where the dot denotes the time derivative. Assuming inextensionality of the beam, and no shear deformation, we can express θ_1 and ψ_1 versus deformations u_1, v_1, w_1 and then angular velocities can be determined from the rotation of the cross section which, after expanding of the trigonometric functions in power series, gives[3],

$$\omega_{\xi_1} = \dot{\phi}_1 + \dot{v}_1' w_1'$$

$$\omega_{\eta_1} = \phi_1 \dot{v}_1' - \dot{w}_1' + \frac{1}{2}\phi_1^2 \dot{w}_1' - \frac{1}{2} w_1'^2 \dot{w}_1' \tag{5}$$

$$\omega_{\zeta_1} = \dot{v}_1' - \frac{1}{2}\phi_1^2 \dot{v}_1' + \frac{1}{2} v_1'^2 \dot{v}_1' + \phi_1 \dot{w}_1' + v_1' w_1' \dot{w}_1'$$

where the prime denotes the space derivative.
Taking into account the geometry of the beam of Fig.2, we can assume that the angular velocity with respect to the η_1 axis, and the mass moment of inertia relative to the ζ_1 axis, are relatively small, therefore,

$$\omega_{\eta_1}^2 \cong 0 ,\ I_{\zeta_1} \cong 0 \ .$$

Thus we keep only that part of the kinetic energy that corresponds to rotation with respect to the ξ_1 axis:

$$\omega_{\xi_1} = \dot{\phi}_1 + \dot{v}_1' w_1' \tag{6}$$

Potential energy is determined in bending about the two principal axes η_1, ζ_1, and torsion about axis ξ_1,

$$V_1 = \frac{1}{2}\int_0^{l_1} \left(D_{\xi_1} \rho_{\xi_1}^2 + D_{\eta_1} \rho_{\eta_1}^2 + D_{\zeta_1} \rho_{\zeta_1}^2 \right) ds_1 \tag{7}$$

where: $D_{\xi_1} = G_1 J_{\xi_1}$ is the torsional stiffness, $D_{\eta_1} = E_1 J_{\eta_1}$, $D_{\zeta_1} = E_1 J_{\zeta_1}$ are flexural stiffnesess and $\rho_{\xi 1}$, $\rho_{\eta 1}$, $\rho_{\eta 1}$ are the curvatures, determined from the angular velocities by using Kirchhoff's kinetic analogue[6]:

$$\rho_{\xi_1} = \phi_1' + w_1' v_1'' \tag{8}$$

$$\rho_{\eta_1} = \phi_1 v_1'' - w_1'' + \frac{1}{2}\phi_1^2 w_1'' - \frac{1}{2} w_1'^2 w_1''$$

$$\rho_{\zeta_1} = v_1'' - \frac{1}{2}\phi_1^2 v_1'' + \frac{1}{2} v_1'^2 v_1'' + \phi_1 w_1'' + v_1' w_1' w_1''$$

The constraint equation for the primary beam has the form,

$$F_1 = \int_0^{l_1} \lambda_1 \left(1 - \left(\left(1 + u_1'\right)^2 + v_1'^2 + w_1'^2\right)\right) ds_1 \tag{9}$$

where λ_1 is the Lagrange multiplier.

The kinetic energy of the secondary beam is calculated by taking into account the velocity $\vec{V}_{O2}$ of the centre of the cross-section O_2, using the radius vector $\vec{r}_2$, related to the translational and angular motions of set $X_2Y_2Z_2$ which has its origin at point *C*, together with the relative velocity $\vec{V}_r$. It can be written in vector form,

$$\vec{V}_{O2} = \vec{V}_C + \vec{\omega}_{C2} \times \vec{r}_2 + \vec{V}_r \tag{10}$$

Applying (10) and then projecting the velocity components onto the $X_2Y_2Z_2$ coordinate set we get the kinetic energy of the secondary beam,

$$T_2 = \frac{1}{2}\int_0^{l_2} \left(\rho_2 A_2 \left(V_{x_2}^2 + V_{y_2}^2 + V_{z_2}^2\right) + I_{\xi_2}\omega_{\xi_2}^2 + I_{\eta_2}\omega_{\eta_2}^2 + I_{\zeta_2}\omega_{\zeta_2}^2\right) ds_2 \tag{11}$$

Absolute velocity components projected onto the moving frame take the forms,

$$\begin{aligned}
V_{x_2} &= \dot{u}_2 + \phi_{1C}\dot{w}_{1C} - v_2\left(\dot{\phi}_{1C} + \dot{v}_{1C}' w_{1C}'\right) - w_2\left(\dot{v}_{1C}' + \phi_{1C}\dot{w}_{1C}\right) + \dot{v}_{1C}\left(1 - \frac{1}{2}\phi_{1C}^2 - \frac{1}{2}v_{1C}'^2\right) \\
&\quad + \dot{u}_{1C}\left(-v_{1C}' - \phi_{1C} w_{1C}'\right) \\
V_{y_2} &= \dot{v}_2 + \left(s_2 + u_2\right)\left(\dot{\phi}_{1C} + \dot{v}_{1C}' w_{1C}'\right) + w_2\left(\phi_{1C}\dot{v}_{1C}' - \dot{w}_{1C}'\right) + \dot{u}_{1C}\left(\phi_{1C} v_{1C}' - w_{1C}'\right) \\
&\quad + \dot{v}_{1C}\left(-\phi_{1C} - v_{1C}' w_{1C}'\right) + \dot{w}_{1C}\left(1 - \frac{1}{2}\phi_{1C}^2 - \frac{1}{2}w_{1C}'^2\right)
\end{aligned} \tag{12}$$

$$V_{z_2} = \dot{w}_2 - (s_2 + u_2)(\dot{v}'_{1C} + \phi_{1C}\dot{w}'_{1C}) + s_2\left(\frac{1}{2}\phi_{1C}^2\dot{v}'_{1C} - \frac{1}{2}\dot{v}'_{1C}v'^2_{1C} - \dot{w}'_{1C}v'_{1C}w'_{1C}\right)$$
$$+ v_2(\phi_{1C}\dot{v}'_{1C} - \dot{w}'_{1C}) + \dot{v}_{1C}v'_{1C} + \dot{w}_{1C}w'_{1C} + \dot{u}_{1C}\left(1 - \frac{1}{2}v'^2_{1C} - \frac{1}{2}w'^2_{1C}\right)$$

And, making assumptions similar to those of the primary beam,

$$\omega_{\eta_2}^2 \cong 0 \quad I_{\zeta_2} \cong 0$$

we get,

$$\omega_{\xi_2} = \dot{\phi}_2 + \dot{v}'_2 w'_2 \tag{13}$$

The potential energy of the secondary beam is expressed in the second local coordinate set, denoted by index *2*, and has an equivalent form to that of the primary beam,

$$V_2 = \frac{1}{2}\int_0^{l_2}\left(D_{\xi_2}\rho_{\xi_2}^2 + D_{\eta_2}\rho_{\eta_2}^2 + D_{\zeta_2}\rho_{\zeta_2}^2\right)ds_2 \tag{14}$$

with torsional and flexural stiffnesses,

$$D_{\xi_2} = G_2 J_{\xi_2}, \quad D_{\eta_2} = E_2 J_{\eta_2}, \quad D_{\zeta_2} = E_2 J_{\zeta_2}$$

and curvatures,

$$\begin{aligned}
\rho_{\xi_2} &= \phi'_2 + w'_2 v''_2 \\
\rho_{\eta_2} &= \phi_2 v''_2 - w''_2 + \frac{1}{2}\phi_2^2 w''_2 - \frac{1}{2}w'^2_2 w''_2 \\
\rho_{\zeta_2} &= v''_2 - \frac{1}{2}\phi_2^2 v''_2 + \frac{1}{2}v'^2_2 v''_2 + \phi_2 w''_2 + v'_2 w'_2 w''_2
\end{aligned} \tag{15}$$

The constraint equation for the secondary beam is defined as,

$$F_2 = \int_0^{l_2}\lambda_2\left(1 - \left((1 + u'_2)^2 + v'^2_2 + w'^2_2\right)\right)ds_2 \tag{16}$$

To obtain the differential equations of motion it is necessary to determine the variations of the functions h_1 and h_2 , L_A, and L_C,

$$\delta h_1 = \sum_{i=1}^{13}\frac{\partial h_1}{\partial p_i}\delta p_i, \quad p = col\left\{\phi_1, \dot{u}_1, \dot{v}_1, \dot{w}_1, \dot{\phi}_1, \dot{v}'_1, u'_1, v'_1, w'_1, \phi'_1, v''_1, w''_1, \lambda_1\right\} \tag{17}$$

$$\delta h_2 = \sum_{i=1}^{25} \frac{\partial h_2}{\partial q_i} \delta q_i \,, \quad \begin{aligned} q = col\{&u_2, v_2, w_2, \phi_2, \dot{u}_2, \dot{v}_2, \dot{w}_2, \dot{\phi}_2, \dot{v}_2', u_2', v_2', w_2', \phi_2', v_2'', w_1'', \\ &\lambda_2, \phi_{1C}, \dot{u}_{1C}, \dot{v}_{1C}, \dot{w}_{1C}, \dot{\phi}_{1C}, v_{1C}', w_{1C}', \dot{v}_{1C}', \dot{w}_{1C}'\} \end{aligned} \tag{18}$$

$$\delta L_C = \sum_{i=1}^{10} \frac{\partial L_C}{\partial p_{Ci}} \delta p_{Ci} \,, \;\; p_C = col\{v_1, \phi_1, \dot{u}_1, \dot{v}_1, \dot{w}_1, \dot{\phi}_1, v_1', w_1', \dot{v}_1', \dot{w}_1'\} \tag{19}$$

$$\delta L_A = \sum_{i=1}^{21} \frac{\partial L_A}{\partial q_{Ai}} \delta q_{Ai} \,, \quad \begin{aligned} q_A = col\{&u_2, v_2, w_2, \phi_2, \dot{u}_2, \dot{v}_2, \dot{w}_2, \dot{\phi}_2, \dot{v}_2', \dot{w}_2', v_2', \\ &w_2', \phi_{1C}, \dot{u}_{1C}, \dot{v}_{1C}, \dot{w}_{1C}, \dot{\phi}_{1C}, v_{1C}', w_{1C}', \dot{v}_{1C}', \dot{w}_{1C}'\} \end{aligned} \tag{20}$$

Integrating the variations by parts with respect to time between limits t_1 and t_2, and remembering that variations at the time instances t_1 and t_2 are equal to zero we get,

$$\begin{aligned} \int_{t_1}^{t_2} \Bigg\{ \int_0^{l_1} \Bigg[&\left(-\frac{\partial^2 h_1}{\partial \dot{u}_1 \partial t} - \frac{\partial^2 h_1}{\partial u_1' \partial s_1} \right) \delta u_1 + \left(-\frac{\partial^2 h_1}{\partial \dot{v}_1 \partial t} + \frac{\partial^3 h_1}{\partial \dot{v}_1' \partial s_1 \partial t} - \frac{\partial^2 h_1}{\partial v_1' \partial s_1} + \frac{\partial^3 h_1}{\partial v_1'' \partial s_1^2} \right) \delta v_1 \\ &+ \left(-\frac{\partial^2 h_1}{\partial \dot{w}_1 \partial t} - \frac{\partial^2 h_1}{\partial w_1' \partial s_1} + \frac{\partial^3 h_1}{\partial w_1'' \partial s_1^2} \right) \delta w_1 + \left(\frac{\partial h_1}{\partial \phi_1} - \frac{\partial^2 h_1}{\partial \dot{\phi}_1 \partial t} - \frac{\partial^2 h_1}{\partial \phi_1' \partial s_1} \right) \delta \phi_1 + \frac{\partial h_1}{\partial \lambda_1} \delta \lambda_1 \Bigg] ds_1 \\ &+ \int_0^{l_2} \Bigg[\left(\frac{\partial h_2}{\partial u_2} - \frac{\partial^2 h_2}{\partial \dot{u}_2 \partial t} - \frac{\partial^2 h_2}{\partial u_2' \partial s_2} \right) \delta u_2 + \left(\frac{\partial h_2}{\partial v_2} - \frac{\partial^2 h_2}{\partial \dot{v}_2 \partial t} + \frac{\partial^3 h_2}{\partial \dot{v}_2' \partial s_2 \partial t} - \frac{\partial^2 h_2}{\partial v_2' \partial s_2} + \frac{\partial^3 h_2}{\partial v_2'' \partial s_2^2} \right) \delta v_2 \\ &+ \left(\frac{\partial h_2}{\partial w_2} - \frac{\partial^2 h_2}{\partial \dot{w}_2 \partial t} - \frac{\partial^2 h_2}{\partial w_2' \partial s_2} + \frac{\partial^3 h_2}{\partial w_2'' \partial s_2^2} \right) \delta w_2 + \left(\frac{\partial h_2}{\partial \phi_2} - \frac{\partial^2 h_2}{\partial \dot{\phi}_2 \partial t} - \frac{\partial^2 h_2}{\partial \phi_2' \partial s_2} \right) \delta \phi_2 + \frac{\partial h_2}{\partial \lambda_2} \delta \lambda_2 \Bigg] ds_2 \Bigg\} dt = 0 \end{aligned} \tag{21}$$

Next integrating by parts with respect to the space coordinates s_1 and s_2, and then collecting terms for proper variations up to the third order, we get successive differential equations of motions:

- ***for the primary beam***

variation δu_1

$$-\rho_1 A_1 \ddot{u}_1 + \lambda_1 u_1'' = 0 \tag{22}$$

variation δv_1

$$
\begin{aligned}
&-\rho_1 A_1 \ddot{v}_1 + \lambda_1 v_1'' + D_{\xi 1}\left(-\phi_1''' w_1' - v_1^{IV} w_1'^2 - w_1'''(\phi_1' + 2w_1' v_1'') - 2w_1''(2v_1''' w_1' + v_1'' w_1'' + \phi_1'')\right) \\
&+D_{\eta 1}\left(w_1^{IV}\phi_1 - v_1^{IV}\phi_1^2 + 2w_1'''\phi_1' - 4v_1'''\phi_1\phi_1' - 2v_1''(\phi_1'^2 + \phi_1\phi_1'') + w_1''\phi_1''\right) \\
&+D_{\zeta 1}\left(-v_1^{IV}\left(1 - \phi_1^2 + v_1'^2\right) - w_1^{IV}\left(\phi_1 + v_1' w_1'\right) + 4v_1'''(\phi_1\phi_1' - v_1' v_1'') - w_1'''(2\phi_1' + w_1' v_1'' + 3v_1' w_1'')\right. \\
&\left.+v_1''(2\phi_1'^2 - v_1''^2 - w_1''^2 + 2\phi_1\phi_1'') - w_1''\phi_1''\right) \\
&+I_1\left(\dot{w}_1'\dot{\phi}_1' + w_1'\left(\ddot{\phi}_1' + 2\dot{w}_1'\dot{v}_1'' + w_1'\ddot{v}_1'' + 2\dot{v}_1'\dot{w}_1''\right) + w_1''\left(\ddot{\phi}_1 + 2w_1'\ddot{v}_1' + 2\dot{v}_1'\dot{w}_1'\right) + \dot{\phi}_1\dot{w}_1''\right) = 0
\end{aligned}
\tag{23}
$$

variation δw_1

$$
\begin{aligned}
&-\rho_1 A_1 \ddot{w}_1 + \lambda_1 w_1'' + D_{\xi 1}\left(v_1'''(\phi_1' + 2w_1' v_1'') + v_1''^2 w_1'' + v_1''\phi_1''\right) \\
&+D_{\eta 1}\left(-w_1^{IV}\left(1 - \phi_1^2 + w_1'^2\right) + v_1^{IV}\phi_1 + 2v_1'''\phi_1' + 4w_1'''(\phi_1\phi_1' - w_1' w_1'') + w_1''\left(2\phi_1'^2 - w_1''^2 + 2\phi_1\phi_1''\right) + v_1''\phi_1''\right) \\
&+D_{\zeta 1}\left(-v_1^{IV}\left(\phi_1 + v_1' w_1'\right) - w_1^{IV}\phi_1^2 - v_1'''(2\phi_1' + 3w_1' v_1'' + v_1' w_1'') - 4w_1'''\phi_1\phi_1' - w_1''\left(2\phi_1'^2 + v_1''^2 + 2\phi_1\phi_1''\right) - v_1''\phi_1''\right) \\
&+I_1\left(-\dot{v}_1'\dot{\phi}_1' - \dot{v}_1'^2 w_1'' - \dot{\phi}_1\dot{v}_1'' - 2w_1'\dot{v}_1'\dot{v}_1''\right) = 0
\end{aligned}
\tag{24}
$$

variation $\delta\phi_1$

$$
\begin{aligned}
&D_{\xi 1}\left(v_1''' w_1' + v_1'' w_1'' + \phi_1''\right) + D_{\eta 1}\left(-\phi_1 v_1''^2 + v_1'' w_1'' + \phi_1 w_1''^2\right) + D_{\zeta 1}\left(\phi_1 v_1''^2 - v_1'' w_1'' - \phi_1 w_1''^2\right) \\
&+I_1\left(-\ddot{\phi}_1 - w_1'\ddot{v}_1' - \dot{v}_1'\dot{w}_1'\right) = 0
\end{aligned}
\tag{25}
$$

variation $\delta\lambda_1$

$$
\left(1 - \left[\left(1 + u_1'\right)^2 + v_1'^2 + w_1'^2\right]\right) = 0
\tag{26}
$$

- ***for the secondary beam***

variation δu_2

$$
\begin{aligned}
&\rho_2 A_2\left(-\ddot{u}_2 - \ddot{v}_{1C}\left(1 - \frac{1}{2}\phi_{1C}^2 - \frac{1}{2}v_{1C}'^2\right) + s_2\left(\dot{\phi}_{1C}^2 + \dot{v}_{1C}'^2 + 2\phi_{1C}\dot{v}_{1C}'\dot{w}_{1C}' + 2\dot{\phi}_{1C}\dot{v}_{1C}' w_{1C}'\right) + u_2\left(\dot{\phi}_{1C}^2 + \dot{v}_{1C}'^2\right)\right. \\
&+v_2\left(\ddot{\phi}_{1C} + \dot{v}_{1C}'^2 + 2\dot{v}_{1C}'\dot{w}_{1C}' + \ddot{v}_{1C}' w_{1C}'\right) + w_2\left(\ddot{v}_{1C}' + \phi_{1C}\ddot{w}_{1C}'\right) + 2\dot{v}_2\left(\dot{\phi}_{1C} + \dot{v}_{1C}' w_{1C}'\right) \\
&\left.+\ddot{u}_{1C}\left(v_{1C}' + \phi_{1C} w_{1C}'\right) - \ddot{w}_{1C}\phi_{1C}\right) + \lambda_2 u_2'' = 0
\end{aligned}
\tag{27}
$$

variation δv_2

$$
\begin{aligned}
&\rho_2 A_2 \left(-\ddot{v}_2 - \ddot{w}_{1C} \left(1 - \frac{1}{2}\phi_{1C}^2 - \frac{1}{2} w_{1C}'^2 \right) - s_2 \left(\ddot{\phi}_{1C} + \phi_{1C} \dot{v}_{1C}'^2 - \phi_{1C} \dot{w}_{1C}'^2 + \ddot{v}_{1C}' w_{1C}' \right) - u_2 \left(\ddot{\phi}_{1C} + \ddot{v}_{1C}' w_{1C}' \right) \right. \\
&\left. + v_2 \left(\dot{\phi}_{1C}^2 + \dot{w}_{1C}'^2 \right) + w_2 \ddot{w}_{1C}' - 2\dot{u}_2 \left(\dot{\phi}_{1C} + \dot{v}_{1C}' w_{1C}' \right) + \ddot{u}_{1C} \left(w_{1C}' - \phi_{1C} v_{1C}' \right) + \ddot{v}_{1C} \left(v_{1C}' w_{1C}' + \phi_{1C} \right) \right) \\
&+ \lambda_2 v_2'' + D_{\xi 2} \left(-\phi_2''' w_2' - v_2^{IV} w_2'^2 - w_2'''(\phi_2' + 2 w_2' v_2'') - 2 w_2'' \left(2 v_2''' w_2' + v_2'' w_2'' + \phi_2'' \right) \right) \\
&+ D_{\eta 2} \left(w_2^{IV} \phi_2 - v_2^{IV} \phi_2^2 + 2 w_2''' \phi_2' - 4 v_2''' \phi_2 \phi_2' - 2 v_2'' \left(\phi_2'^2 + \phi_2 \phi_2'' \right) + w_2'' \phi_2'' \right) \\
&+ D_{\zeta 2} \left(-v_2^{IV} \left(1 - \phi_2^2 + v_2'^2 \right) - w_2^{IV} \left(\phi_2 + v_2' w_2' \right) + 4 v_2''' (\phi_2 \phi_2' - v_2' v_2'') - w_2''' (2\phi_2' + w_2' v_2'' + 3 v_2' w_2'') \right. \\
&\left. + v_2'' \left(2\phi_2'^2 - v_2''^2 - w_2''^2 + 2\phi_2 \phi_2'' \right) - w_2'' \phi_2'' \right) \\
&+ I_2 \left(\dot{w}_2' \dot{\phi}_2' + w_2' \left(\ddot{\phi}_2' + 2 \dot{w}_2' \dot{v}_2'' + w_2' \ddot{v}_2'' + 2 \dot{v}_2' \dot{w}_2'' \right) + w_2'' \left(\ddot{\phi}_2 + 2 w_2' \ddot{v}_2' + 2 \dot{v}_2' \dot{w}_2' \right) + \dot{\phi}_2 \dot{w}_2'' \right) = 0
\end{aligned}
\tag{28}
$$

variation δw_2

$$
\begin{aligned}
&\rho_2 A_2 \left(-\ddot{w}_2 + s_2 \left(\ddot{v}_{1C}' \left(1 - \frac{1}{2}\phi_{1C}^2 + \frac{1}{2} v_{1C}'^2 \right) + \ddot{w}_{1C}' \left(\phi_{1C} + v_{1C}' w_{1C}' \right) + v_{1C}' \left(\dot{v}_{1C}'^2 + \dot{w}_{1C}'^2 \right) \right) + u_2 \left(\ddot{v}_{1C}' + \phi_{1C} \ddot{w}_{1C}' \right) \right. \\
&+ v_2 \left(\ddot{w}_{1C}' - \phi_{1C} \ddot{v}_{1C}' \right) + w_2 \left(\dot{v}_{1C}'^2 + \dot{w}_{1C}'^2 \right) + 2\dot{u}_{1C} \left(\dot{v}_{1C}' v_{1C}' + \dot{w}_{1C}' w_{1C}' \right) - 2\dot{v}_{1C} \dot{v}_{1C}' - 2\dot{w}_{1C} \dot{w}_{1C}' \\
&\left. - \ddot{u}_{1C} \left(1 - \frac{1}{2} v_{1C}'^2 - \frac{1}{2} w_{1C}'^2 \right) - \ddot{v}_{1C} v_{1C}' - \ddot{w}_{1C} w_{1C}' \right) + \lambda_2 w_2'' + D_{\xi 2} \left(v_2''' (\phi_2' + 2 w_2' v_2'') + v_2''^2 w_2'' + v_2'' \phi_2'' \right) \\
&+ D_{\eta 2} \left(-w_2^{IV} \left(1 - \phi_2^2 + w_2'^2 \right) + v_2^{IV} \phi_2 + 2 v_2''' \phi_2' + 4 w_2''' (\phi_2 \phi_2' - w_2' w_2'') + w_2'' \left(2\phi_2'^2 - w_2''^2 + 2\phi_2 \phi_2'' \right) + v_2'' \phi_2'' \right) \\
&+ D_{\zeta 2} \left(-v_2^{IV} \left(\phi_2 + v_2' w_2' \right) - w_2^{IV} \phi_2^2 - v_2''' (2\phi_2' + 3 w_2' v_2'' + v_2' w_2'') - 4 w_2''' \phi_2 \phi_2' - w_2'' \left(2\phi_2'^2 + v_2''^2 + 2\phi_2 \phi_2'' \right) - v_2'' \phi_2'' \right) \\
&+ I_2 \left(-\dot{v}_2' \dot{\phi}_2' - \dot{v}_2'^2 w_2'' - \dot{\phi}_2 \dot{v}_2'' - 2 w_2' \dot{v}_2' \dot{v}_2'' \right) = 0
\end{aligned}
\tag{29}
$$

variation $\delta\phi_2$

$$
\begin{aligned}
&D_{\xi 2} \left(v_2''' w_2' + v_2'' w_2'' + \phi_2'' \right) + D_{\eta 2} \left(-\phi_2 v_2''^2 + v_2'' w_2'' + \phi_2 w_2''^2 \right) + D_{\zeta 2} \left(\phi_2 v_2''^2 - v_2'' w_2'' - \phi_2 w_2''^2 \right) \\
&+ I_2 \left(-\ddot{\phi}_2 - w_2' \ddot{v}_2' - \dot{v}_2' \dot{w}_2' \right) = 0
\end{aligned}
\tag{30}
$$

variation $\delta\lambda_2$

$$\left(1-\left[\left(1+u_2'\right)^2+v_2'^2+w_2'^2\right]\right)=0 \tag{31}$$

The components obtained from integration by parts for the limits $s_1=0$, $s_1=l_1$ and $s_2=0$, $s_2=l_2$, and then grouped for the appropriate variations, give the associated boundary conditions:

- *at point B,* $s_1=0$

$$u_{1B}=0,\ v_{1B}=0,\ w_{1B}=0,\ \phi_{1B}=0,\ v_{1B}'=0,\ w_{1B}'=0 \tag{32}$$

- *at point C,* $s_1=l_1,\ s_2=0$

variation δu_{1C}

$$-\lambda_1\left(1+u_{1C}'\right)-\rho_2A_2l_2\left[\ddot{w}_{2A}+\ddot{u}_{1C}-l_2\ddot{v}_{1C}'\right]-m_C\ddot{u}_{1C}-m_A\left[\ddot{w}_{2A}+\ddot{u}_{1C}-l_2\ddot{v}_{1C}'\right]+HOT=0 \tag{33}$$

variation δv_{1C}

$$\left(D_{\zeta1}v_{1C}'''-\lambda_1v_{1C}'\right)-\rho_2A_2l_2\left[\ddot{u}_{2A}+\ddot{v}_{1C}\right]-m_Cg-m_C\ddot{v}_{1C}-m_Ag\phi_{1C}-m_A\left[\ddot{u}_A+\ddot{v}_{1C}\right]+HOT=0 \tag{34}$$

variation δw_{1C}

$$\left(D_{\eta1}w_{1C}'''-\lambda_1w_{1C}'\right)-\rho_2A_2l_2\left[\ddot{v}_{2A}+\ddot{w}_{1C}+l_2\ddot{\phi}_{1C}\right]-m_C\ddot{w}_{1C}-m_A\left[\ddot{v}_{2A}+\ddot{w}_{1C}+l_2\ddot{\phi}_{1C}\right]+HOT=0 \tag{35}$$

variation $\delta\phi_{1C}$

$$\begin{gathered}\left(-D_{\xi1}\phi_{1C}'\right)-\rho_2A_2l_2\left[l_2\left(\ddot{v}_{2A}+\ddot{w}_{1C}\right)+l_2^2\ddot{\phi}_{1C}\right]-I_{C\xi}\ddot{\phi}_{1C}-m_A\left[l_2\left(\ddot{v}_{2A}+\ddot{w}_{1C}\right)+l_2^2\ddot{\phi}_{1C}\right]\\+m_Ag\left(v_{2A}+l_2\phi_{1C}\right)+HOT=0\end{gathered} \tag{36}$$

variation $\delta v_{1C}'$

$$\begin{gathered}\left(-D_{\zeta1}v_{1C}''\right)-\rho_2A_2l_2\left[-l_2\left(\ddot{w}_{2A}+\ddot{u}_{1C}\right)+l_2^2\ddot{v}_{1C}'\right]-I_{C\zeta}\ddot{v}_{1C}'\\-m_A\left[-l_2\left(\ddot{w}_{2A}+\ddot{u}_{1C}\right)+l_2^2\ddot{v}_{1C}'\right]-m_Ag\left(w_{2A}-l_2v_{1C}'\right)+HOT=0\end{gathered} \tag{37}$$

variation $\delta w_{1C}'$

$$\left(-D_{\eta1}w_{1C}''\right)-I_{C\eta}\ddot{w}_{1C}'+HOT=0 \tag{38}$$

- *at point A,* $s_2 = l_2$

variation δu_{2A}

$$-\lambda_2\left(1+u'_{2A}\right)l_2 - m_A g - m_A\left[\ddot{u}_{2A} + \ddot{v}_{1C}\right] + HOT = 0 \tag{39}$$

variation δv_{2A}

$$\left(D_{\zeta 2}v''''_{2A} - \lambda_2 v'_{2A}\right) - m_A g\phi_{1C} - m_A\left[\ddot{v}_{2A} + \ddot{w}_{1C} + l_2\ddot{\phi}_{1C}\right] + HOT = 0 \tag{40}$$

variation δw_{2A}

$$\left(D_{\eta 2}w''''_{2A} - \lambda_2 w'_{2A}\right) - m_A g v'_{1C} - m_A\left[\ddot{w}_{2A} + \ddot{u}_{1C} - l_2\ddot{v}'_{1C}\right] + HOT = 0 \tag{41}$$

variation $\delta\phi_{2A}$

$$\left(-D_{\xi 2}\phi'_{2A}\right) - I_{A\xi}\ddot{\phi}_{2A} + HOT = 0 \tag{42}$$

variation $\delta v'_{2A}$

$$\left(-D_{\zeta 2}v''_{2A}\right) - I_{A\zeta}\ddot{v}'_{2A} + HOT = 0 \tag{43}$$

variation $\delta w'_{2A}$

$$\left(-D_{\eta 2}w''_{2A}\right) - I_{A\eta}\ddot{w}'_{2A} + HOT = 0 \tag{44}$$

Formulae for boundary conditions are given up to the first order terms while the second and third orders are written by the abbreviation *HOT,* that means higher order terms. Indexes *A*, *B*, *C* denote values at proper points. Note that to have consistency in Eqs. (33)-(38) variations of the secondary beam at point $s_2=0$ are expressed by variations of the primary beam at $s_1=l_1$, by using a transformation of the local to the absolute set of coordinates.

The derived partial differential equations which describe the problem consist of the geometrical and inertial nonlinear terms and nonlinear, non-homogenous, dynamical boundary conditions. To solve this set of nonlinear equations of motion, and the nonlinear boundary conditions, an approximate solution method has to be applied. It requires an appropriate assumption for the admissible vibration modes which will then satisfy the boundary conditions to the required perturbation order accuracy. This will be completed in further analytical investigations of this problem, to be undertaken in the very near future. However, to make proper assumptions for this further work on an approximate analytical approach experimental and numerical (FEA) tests have been undertaken and these are presented in the next section.

4 EXPERIMENT

The experimental setup used for the testing work is composed of a high end proprietary modal analysis system, spectral acquisition software, and an electrically matched shaker with feedback loop control of the excitation level. The signals are measured by three small, low mass, piezo-sensors and a piezo-sensor is used for monitoring the excitation. The arrows in Fig. 1 indicate the orientations of the sensors used in the experimental tests.

Preliminary experimental investigations consisted of tuning the structure for chosen bending and torsional natural frequencies of the system. The frequencies are determined by modal analysis of the system response activated by an impact. By modification of lumped masses *A* and *C* and the length of the primary beam, the system has been tuned for a 1:4 ratio of the first bending frequency of the primary beam $\left(\omega_{b1(I)} = 3.61 Hz\right)$ and the first bending frequency of the secondary beam $\left(\omega_{b1(II)} = 14.45 Hz\right)$. The torsional frequency of the primary beam, when the whole structure is fixed, has also been measured $\left(\omega_{t1(I)} =\sim 4.9 Hz\right)$. The parameters of the tuned structure are listed in Table 1.

length of horizontal beam	236 mm
length of vertical beam	201 mm
mass A value	15.3 gr
mass C value	38.0 gr

Table 1. Parameters for structure after tuning 1:4

After the tuning procedure, the whole structure is mounted on the shaker and then excited by a random excitation over the band from 0 to 40 Hz. This test enables the resonant responses of the system to be found. Figures 3 and 4 show, respectively, the frequency spectra of the system responses obtained from sensors No.1 (Z direction) and No.3 (Y direction).

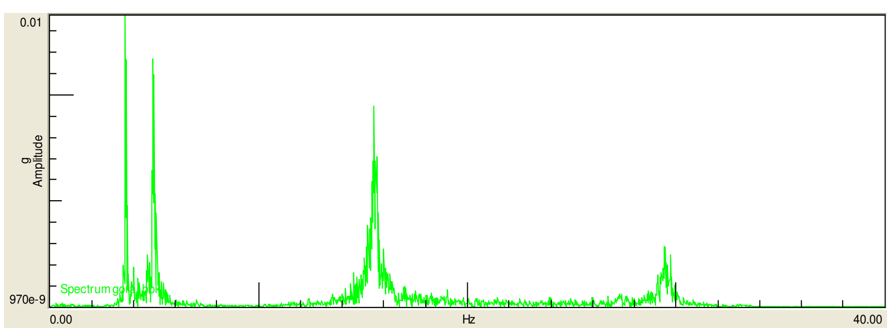

Figure 3. Spectrum of the response measured by sensor No. 1

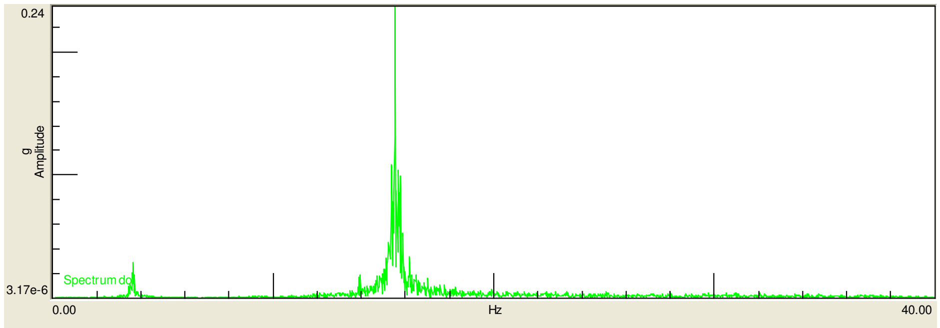

Figure 4. Spectrum of the response measured by sensor No. 3

The highest peaks in Fig.3 correspond to the resonant frequencies of the system, which in a linear system are equal to the natural frequencies. Individual peaks, in turn, correspond to the first bending frequency $\omega_{b1(I)} = 3.61Hz$, the first torsional frequency $\omega_{t1(I)} = 4.9Hz$, and the second free bending frequency $\omega_{b2(I)} = 15.50Hz$, of the primary beam. Because of the positioning of the sensor, Fig.4 mainly shows the dynamics of the primary beam. The closeness of the torsion and bending frequencies is a typical feature of such a structure when made of composite material. For geometrically equivalent aluminium or steel beams, the torsion natural frequency would tend to be remote from the bending frequencies.

The structure has been also modelled in the ABAQUS finite element package by using laminated (layered) shell elements. Results based on linear modal analysis of the FEM model and those obtained experimentally are compared in Table 2.

	Physical model	FEM model
Fig. 5a	3.61 Hz	3.78 Hz
Fig. 5b	4.90 Hz	4.21 Hz
Fig. 5c	15.50 Hz	16.10 Hz
Fig. 5d	29.60 Hz	29.25 Hz

Table 2. Comparison experiment and FEM results.

Vibration modes which correspond to the frequencies presented in Table 2 are shown in Fig.5. For better visualisation the modes of vibrations are plotted together with the undeformed structure. The first mode with the lowest frequency value, that is 3.61 Hz (the first peak in Fig.3 and Fig.4) is given in Fig.5a, evidently the first bending mode of the primary beam is responsible for the dynamics. The vertical beam moves in the vertical plane as a solid body. The mode at 4.9 Hz represents torsion of the primary beam (Fig.5b, the second peak in Fig.3) while the mode at 15.5 Hz corresponds to the second bending mode of the primary beam (Fig.6c and the third peak in Fig.3 and Fig.4). It is worth noting that the three lowest vibration modes, which have been separated by linear modal analysis, only exhibit deformations in bending and torsion of the primary beam. The secondary beam, which has the same cross-section, remains undeformed. An interesting phenomenon has also been observed by studying the fourth peak of Fig.3 in detail.

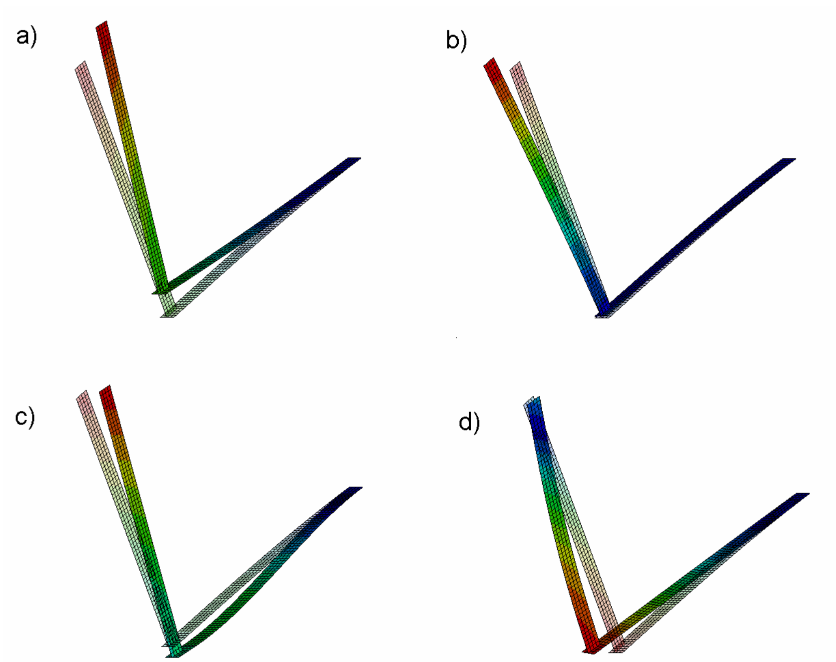

Figure 5. Linear modes of the structure a) the first bending mode of the primary beam, b) the first torsional mode of the primary beam, c) the second bending mode of the primary beam, d) the first bending mode of the primary beam in the stiffer direction coupled with bending mode of the secondary beam.

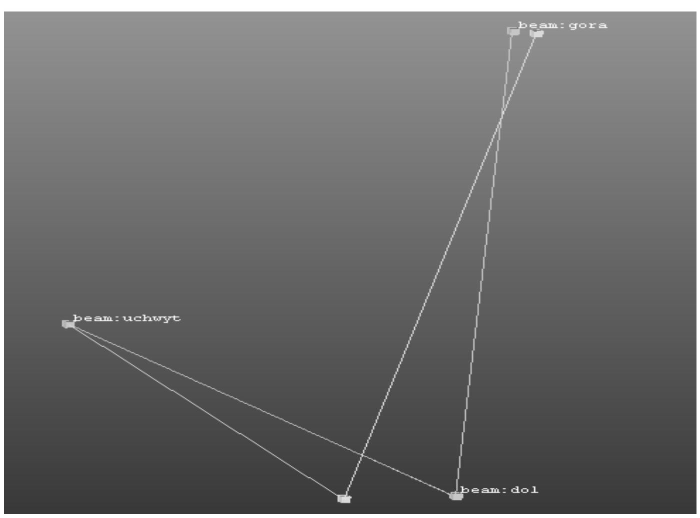

Figure 6. Experimental modal analysis (equivalent to Fig. 5d).

For this frequency the bending mode of the vertical beam is excited, and, due to interaction, the bending mode of the horizontal beam in the stiff direction is excited too. This out-of-plane motion is presented in Fig.5d and is confirmed experimentally in Fig.6.

In many practical engineering applications, the control of motion of the top mass A plays an important role. Therefore, the influence of the internal resonance conditions on trajectories at this point is of interest. By imposing harmonic excitations at different frequencies, in particular around the resonant areas, the response of the system can be investigated in some detail. In this paper we only consider the resonant response around the torsional frequency of the primary beam. This resonance has not been taken into account in the literature to date, to the authors' knowledge. To avoid damage to the structure, and to get satisfactory signals, the amplitude of excitation has been carefully chosen. Fig.7 shows trajectories of the top mass near the torsional resonance of the primary beam. The trajectories are reconstructed by signals received from sensors No.1 and 2.

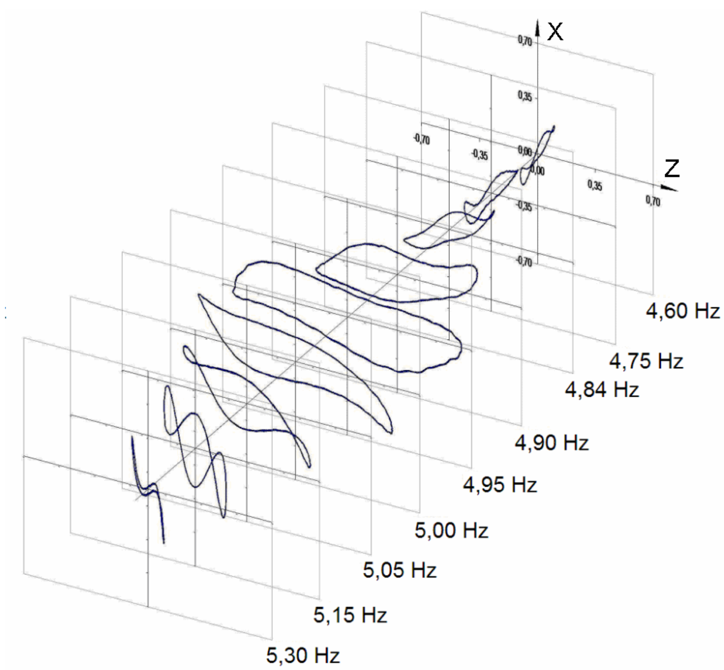

Figure 7. Trajectories of the top mass.

During transition through the resonance, differences in the structural response are clearly visible. Inside the resonance area, near 4.9 Hz, the major axis of an elliptic trajectory is almost parallel to the Z coordinate. Outside this resonance zone the axis rotates in the clockwise direction and the trajectory, because of nonlinear interactions with other vibrations modes, assumes a more complex shape, reminiscent of a Lissajous figure.

5 CONCLUSIONS AND FINAL REMARKS

The paper deals with preliminary theoretical and well developed experimental studies of an autoparametric beam structure with essentially different stiffnesses in two orthogonal directions. The systematically derived equations of motion, and the associated dynamical boundary conditions, show that nonlinear terms which couple the structure may result in many unexpected responses. An experimentally tested composite beam structure, tuned for the 1:4 internal resonance condition, exhibits possible vibrations as an out-of-plane motion in the stiff direction of the primary beam. In the neighbourhood of the torsional resonance, due to nonlinear coupling, additional nonlinear modes are involved in the system response, and this is expressed by the complex trajectories that have been seen. The experimental work has confirmed the FEA analysis, with generally very good agreement. Therefore the results give a promising basis for finding and interpreting analytical solutions of the mathematical model, as summarised in section 3. This, and further investigation will eventually allow a strategy to be developed for the active control of this kind of structure by the application of PZT or SMA elements.

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