Substitute acoustic boundary impedance conditions for boundaries with periodic geometry in computer simulations of acoustic planar wave traveling

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Abstract

In the paper the substitute acoustic boundary impedance conditions for boundaries with periodic geometry are considered. The complex geometry of boundary is treated as planar one with the equivalent impedance conditions obtained from the process of numerical homogenization. The computer simulations used the finite element method (FEM) and all analyses took into consideration the sinusoidal acoustic planar waves. Homogenized boundaries, significantly decreases number of degrees of freedom in acoustic analysis, what in many cases, is the only way to overcome extremely high hardware requirements [7].

1 Introduction

In the numerical analysis that consider acoustic wave traveling phenomena in finite domain, the boundaries play the crucial role. The wave reflection, absorption or transmission on boundaries have significant influence on acoustic pressure distribution, so it is important to apply appropriate boundary impedance conditions. It is obvious that the absorptive properties of acoustic materials are not only caused by physical properties itself. The geometry of boundaries plays crucial role in the process of acoustic wave energy dissipation thus the great notice must be laid on this aspect.

In Fig. (1) the influence of corrugation on absorption properties of acoustic material is shown [1]. It is clearly seen that the corrugation of boundary has changed the characteristics of tested acoustic material even up to 25% (for the higher frequencies). This phenomena can be explained theoretically and it will be discussed further.

The exact mapping of complex boundaries can result in a great number of additional variables in the model. In the paper, we show how for the special cases of boundaries (the periodic ones) such situations can be avoided by the process of boundary homogenization (see Fig. (2)). The introduction of substitute boundary impedance condition allows us to treat the corrugated boundaries as planar one, but with some restrictions discussed later.

2 Reflection from corrugated surface

2.1 Theory

Let us introduce the scalar function $\Phi = \Phi(t)$ of an acoustic potential (velocity potential) [3], so

$$\mathbf{v} = -grad\Phi, \tag{1}$$

where t is time and $\mathbf{v}$ is vector of the acoustic particle velocity.

[1]experiment tests were done by using Kundt's Tube

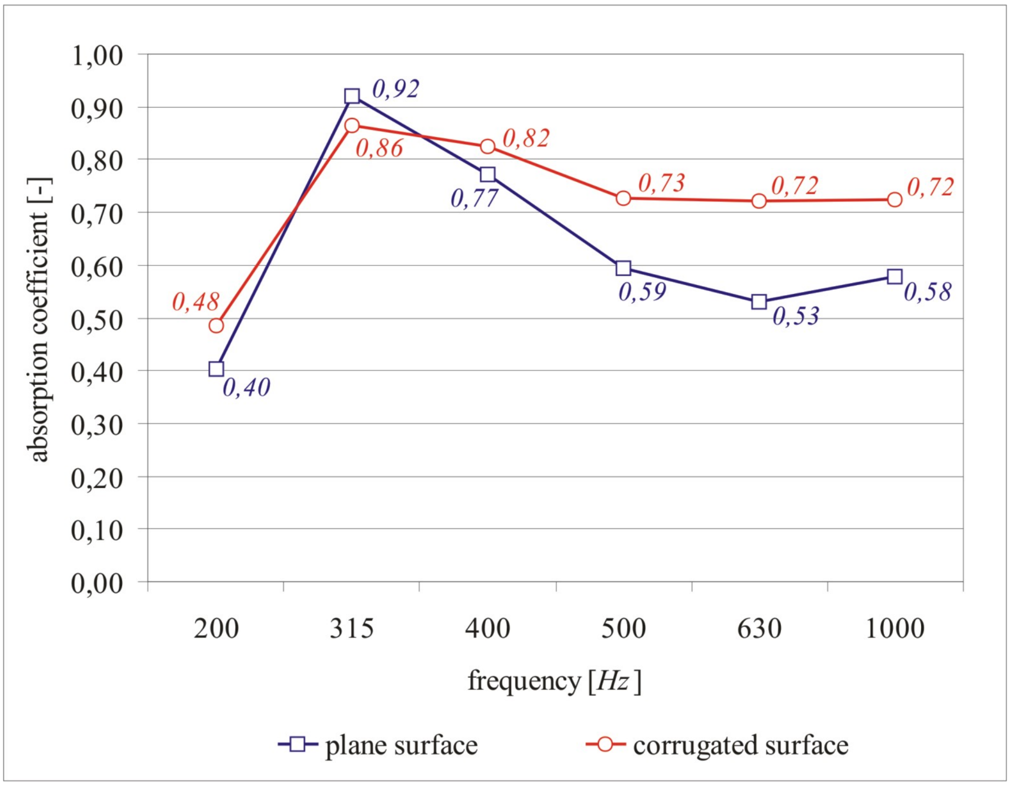

Figure 1: The comparison of absorption coefficient distribution between two identical acoustic materials with plane surface and corrugated surface

Using the function Φ the equilibrium equation for acoustic domain (wave equation) can be written as

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}, \tag{2}$$

where c is the wave speed and ∇^2 is the Laplace operator.

The wave equation in finite domain has to be satisfied together with boundary conditions. At the boundary of two different media the following conditions arises:

1. the stream continuity conditions must be satisfied, thus the velocities of acoustic particles normal to the boundary have to be identical, so

$$\frac{\partial\Phi_1}{\partial\mathbf{n}} = \frac{\partial\Phi_2}{\partial\mathbf{n}}, \tag{3}$$

   where vector $\mathbf{n}$ represents the normal to the acoustic medium at the boundary, Φ_1 and Φ_2 are the resultant fields in first and second medium respectively,

2. the acoustic pressure at both boundaries has be equal, thus

$$\rho_1\Phi_1 = \rho_2\Phi_2, \tag{4}$$

   where ρ_i is density of i-th fluid.

The level of absorption of acoustic wave energy at boundary can be defined in terms of acoustic impedance Z, i.e. the ratio of acoustic pressure p and the velocity of acoustic particle $\mathbf{v}$. Thus from the definition, the acoustic impedance Z, in general form, can be written as

$$Z = \rho_0\frac{\partial\Phi}{\partial t}(-\mathrm{grad}\Phi)^{-1}, \tag{5}$$

where ρ_0 is the density of the fluid and the formula $p = \rho_0\frac{\partial\Phi}{\partial t}$ was used.

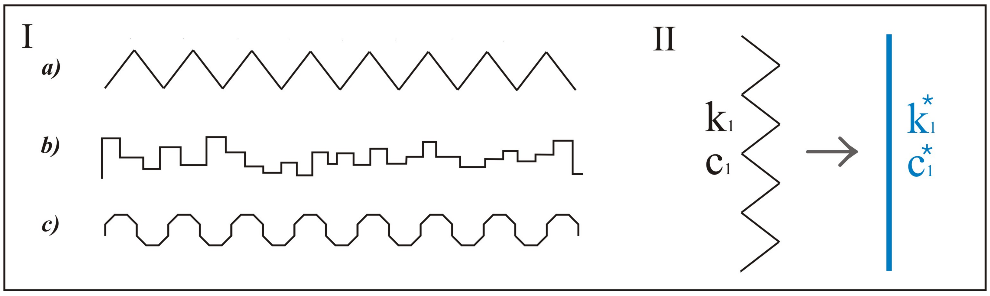

Figure 2: I) The cross sections through a typical acoustic materials: a) pyramid, b) rectangular prism, c) other. II) The scheme of homogenization idea

2.2 Example

2.2.1 Introduction

The following example has the crucial meaning for the analyses of reflection of acoustic wave from boundary with periodic geometry. Obviously, all periodic functions can be approximated by trigonometric Fourier series. Thus if the solution for sinusoidal corrugation is known the solution for any other periodic corrugation can be obtained by summation of successive terms of Fourier series (the sinusoidal one).

The considered problem is shown in Fig.(3). The acoustic planar wave travels parallel to the y axis in the direction of positive coordinates. The corrugation of boundary has sinusoidal shape thus it can be described by the function

$$y = D\cos(hx), \tag{6}$$

where D is the amplitude and $\lambda = \frac{2\pi}{h}$ is wavelength of corrugation . The homogenized boundary has to guar-

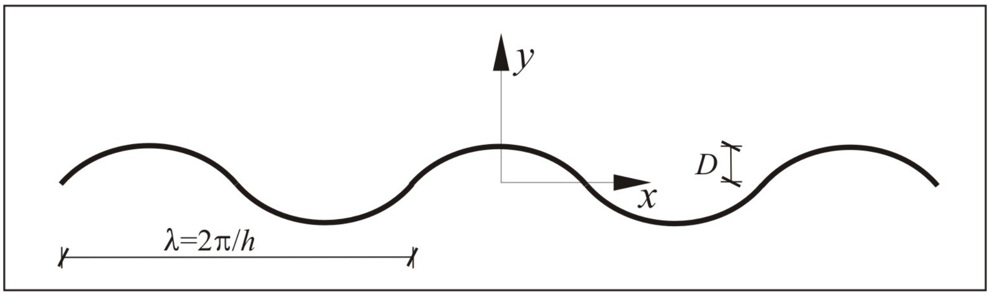

Figure 3: The scheme of the problem

antee that the acoustic pressure field is not changed considerably. That is why we have to use the assumption which has the fundamental importance. The corrugation has to be “flat”, i.e. the following condition must be fulfilled

$$D \ll \frac{2\pi}{h}. \tag{7}$$

Such a strong assumption additionally eliminates the possibility of internal reflections (inside the folding) and makes the analytical solution possible to obtain.

2.2.2 Solution

The incident wave can be described by the following formula

$$\Phi_1' = |\Phi_1'| \exp[-i(k_1 y + \omega t)], \tag{8}$$

where ω is the frequency $[\frac{rad}{s}]$, $k_1 = \frac{\omega}{c_1}$ is the wave number and $i = \sqrt{-1}$ is the imaginary number. The reflected wave consists of a series of packages reflected under various angles, dependent on the point in which the package has reached the boundary. Thus the reflected wave (in first medium) can be obtained from Fourier series (the following harmonics of corrugation) as

$$\Phi_1'' = A_0 \exp[-ik_1 y] + A_1 \exp[-iq_1 y]\cos(hx) + A_2 \exp[-iq_2 y]\cos(2hx) + \dots, \tag{9}$$

where $q_m = \sqrt{k^2 - (mh)^2}$, $m = 1, 2, 3, \dots$ are the wave numbers of the following components, $cos(mhz)$ represents the phase shift due to differences in way needed to reach the boundary by package and $A_0, A_1, \dots, A_m$ are amplitudes of the following components. The coefficient A_0 represents the *principal reflected wave* (reflection from the plane) and the rest A_m factors has different physical meaning depending on if their wave numbers q_m are real or imaginary. When the wave number is real ($k_1 > mh$) it means that the package is simply the common homogenous wave and when the wave number is imaginary ($k_1 < mh$) we obtain inhomogeneous wave exponentially damped perpendicular to boundary (they can be omitted in computations) [3].

From the information given above the following conclusions of crucial meaning arises:

1. if $\lambda_c < \lambda_w$ only the *principal reflected wave* is observed after reflection,
2. if $\lambda_c > \lambda_w$ except the *principal reflected wave* other component of expansion in Eqn. (9) are important,

   λ_c and λ_w are the wavelength of corrugation and wave, respectively.

In further part of the paper, we take into consideration the waves and corrugations that satisfies the conditions from the first point. Such assumption guarantee that the acoustic pressure field will not change much due to homogenization of boundaries. It will be shown that in limit, i.e. when $\lambda_c \ll \lambda_w$ the corrugation don't influence on acoustic pressure field.

By using the assumption from Eqn. (7) the boundary condition described by Eqn. (3) can be rewritten in the form

$$\frac{\partial \Phi_1}{\partial y} = \frac{\partial \Phi_2}{\partial y}, \tag{10}$$

and for the fist terms of Eqn. (9) we can assume $q_m \approx k_1$. Thus the resultant wave in the first medium is

$$\Phi_1 = \exp[ik_1 y] + A_0 \exp[-ik_1 y] + A_1 \exp[-ik_1 y]\cos(hx) + A_2 \exp[-ik_1 y]\cos(2hx) + \dots, \tag{11}$$

and its first derivative

$$\frac{\partial \Phi_1}{\partial y} = ik_1\{\exp[ik_1 y] - A_0 \exp[-ik_1 y] - A_1 \exp[-ik_1 y]\cos(hx) - A_2 \exp[-ik_1 y]\cos(2hx) - \dots\}. \tag{12}$$

The resultant wave in the second medium can be written as

$$\Phi_2 = B_0 \exp[ik_2 y] + B_1 \exp[iq_1' y]\cos(hx) + B_2 \exp[iq_2' y]\cos(2hx) + \dots, \tag{13}$$

so using Eqns. (3) and (4) we obtain the following formula

$$\frac{k_2\rho_1 - k_1\rho_2}{k_2\rho_1 + k_1\rho_2} \exp[2ik_1 y] + A_0 + A_1\cos(hx) + A_2\cos(2hx) + \dots = 0, \tag{14}$$

or in terms of acoustic impedance

$$-\frac{Z_2 - Z_1}{Z_2 + Z_1} \exp[2iky] + A_0 + A_1\cos(hx) + A_2\cos(2hx) + \dots = 0, \tag{15}$$

where $Z_i = \rho_i c_i$ is an acoustic impedance of *i-th* medium and $b = \frac{Z_2 - Z_1}{Z_2 + Z_1}$ is the *reflection indicator*. Let us notice, if we put $y = 0$ the coefficients $A_1, \dots, A_m$ vanish, while

$$A_0 = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \tag{16}$$

expressing the amplitude of the wave reflected from plane surface.

The Jacobi-Anger expansion of function $\exp[ia\cos(nx)]$ defined as

$$\exp[ia\cos(nx)] = J_0(a) + 2\sum_{m=1}^{\infty} i^m J_m(a)\cos(mnx), \tag{17}$$

where J_m are the Bessel functions of the first kind, shows that the Eqn. (15) will be satisfy as identity if coefficients A_m would be, in general, as follows ($k_1 = k$)

$$\begin{aligned} A_0 &= bJ_0(2kD), \\ A_m &= 2(-i)^m bJ_m. \end{aligned} \tag{18}$$

If the condition $\lambda_c < \lambda_w$ is fulfilled we can assume that only the *principal reflected wave* exists after reflection with $A_0 = bJ_0(2kD)$ (Eqn. (18)). The comparison with Eqn. (16) indicates how the corrugation reduces the amplitude of reflected wave - the multiplier $J_0(2kD)$ allows us to estimate the new value of acoustic impedance for homogenized boundary. Let us noticed that in limit, i.e. when $\lambda_c \ll \lambda_w$ (or $2kD \to 0^+$) the corrugation do not influence on the acoustic pressure field due to the fact that in the limit the multiplier

$$\lim_{2kD\to 0^+} J_0(2kD) = 1.$$

2.2.3 Computational example

Let us compute the reflected acoustic wave from the sinusoidal surface and the substitute impedance conditions for homogenized boundaries (see Fig.(2)). The incident wave has the frequency $f = 340Hz$ thus $\lambda_w = 100cm$ ($k = 0.063cm^{-1}$). The wavelength of corrugation is $\lambda_c = 50cm$ ($h = 0.126cm^{-1}$) and the amplitude of corrugation is $D = 2cm$. The *reflection indicator* b is assumed to be one.

Solution based on theory from Section 2.2.2 [2]

According to the previous explanations, because the assumption that $\lambda_c = 50cm < \lambda_w = 100cm$ is fulfilled, we can consider that the reflected wave consists of the *principal reflected wave* only. The reflected wave can be then expressed as follows (see Eqn. (9))

$$\Phi_1'' = A_0 \exp[-iky], \tag{19}$$

where $A_0 = bJ_0(2kD) = 0.984$. We would get identical solution if we consider that the boundary is plane, and that the solution from Eqn. (16) is correct but if the impedance Z_2 is substitute by

$$Z_2^* = Z_1 \frac{A_0^p J_0(2kD) + 1}{1 - A_0^p J_0(2kD)}, \tag{20}$$

where $A_0^p = \frac{Z_2 - Z_1}{Z_2 + Z_1}$.

FEM solution [3]

The solution of the acoustic problems in commercial FEM systems is based on generalized form of wave equation (see Eqn. (2)) (the internal friction is included) [1, 6]

$$\frac{\partial p}{\partial \mathbf{x}} + \gamma(\mathbf{x}, \theta_i)\mathbf{v} + \rho(\mathbf{x}, \theta_i)\dot{\mathbf{v}} = 0, \tag{21}$$

where p is the excess pressure in the fluid (the pressure in excess of any static pressure), $\mathbf{x}$ is the spatial position of the fluid particle, $\dot{\mathbf{v}}$ is the fluid particle acceleration, γ is the "volumetric drag" (force per unit volume per velocity), and θ_i are i independent field variables such as temperature, humidity of air, or salinity of water

[2] computations were done in a free scientific software package for numerical computations *SciLab 3.1.1*

[3] computations were done in a commercial FE system ABAQUS v6.5

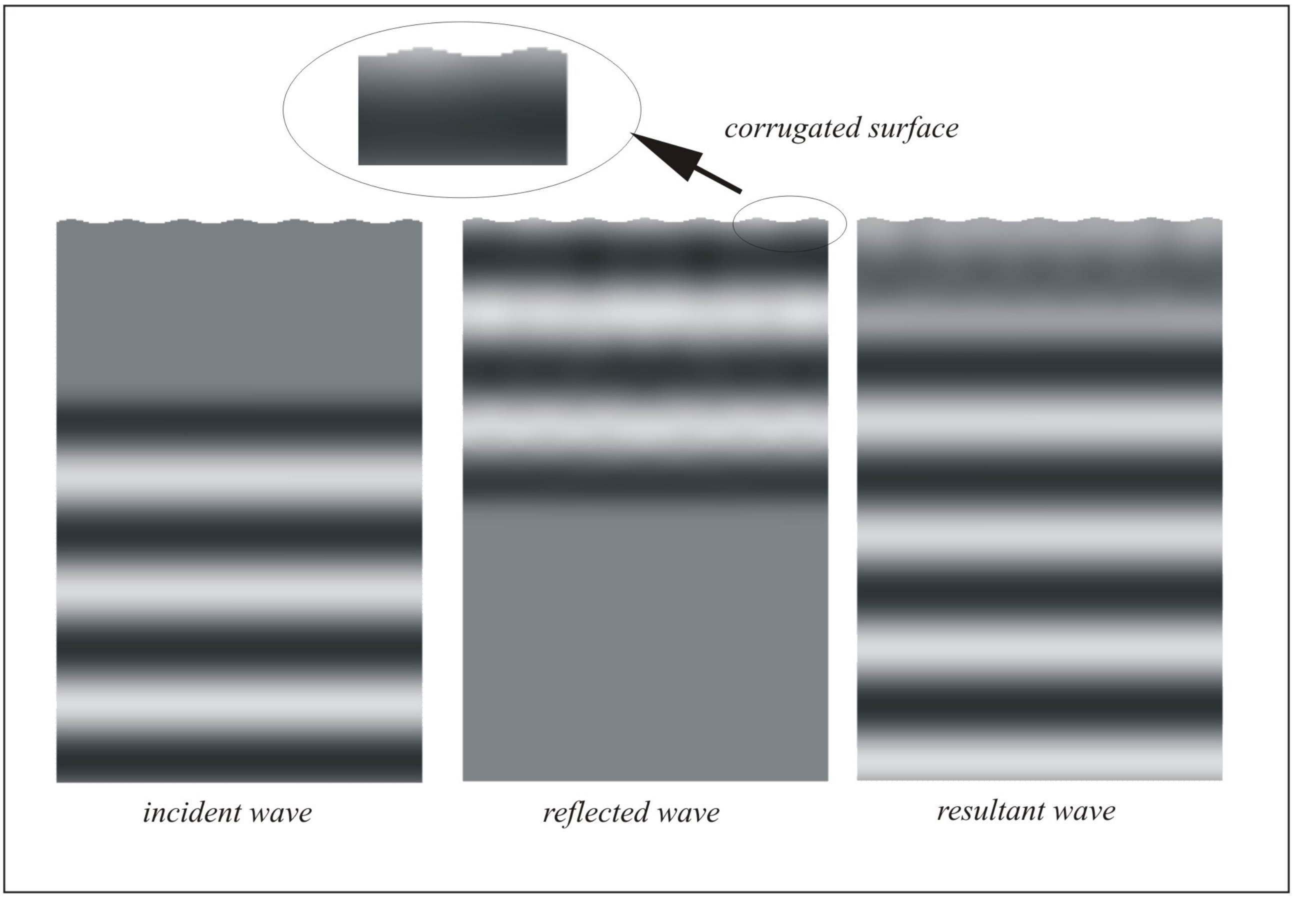

Figure 4: The results obtained from numerical analyses

on which ρ and γ may depend.The constitutive behavior of the fluid is assumed to be inviscid, linear and compressible, so

$$p = -K(\mathbf{x}, \theta_i)\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{u}, \tag{22}$$

where K is the bulk modulus of the fluid and $\mathbf{u}$ is the fluid particle displacement.

The final FE equations are obtained from the weak form of Eqn. (21) after substituting the boundary terms[4], thus

$$\int_V [\delta p(\frac{1}{K}\ddot{p} + \frac{\gamma}{\rho K}\dot{p}) + \frac{1}{\rho}\frac{\partial}{\partial \mathbf{x}}\delta p \cdot \frac{\partial}{\partial \mathbf{x}} p]dV + \int_{S_{fr}} \delta p(\frac{\gamma}{\rho}\frac{1}{c_1}p + (\frac{\gamma}{\rho}\frac{1}{k_1} + \frac{1}{c_1})\dot{p} + \frac{1}{k_1}\ddot{p})dS +$$
$$\int_{S_{fi}} \delta p(\frac{1}{c_1}\dot{p} + \frac{1}{a_1}p)dS - \int_{S_{fs}} \delta p(\mathbf{n} \cdot \dot{\mathbf{v}}^m)dS - \int_{S_{ft}} \delta p(\mathbf{n} \cdot \dot{\mathbf{v}})dS +$$
$$\int_{S_{frs}} \delta p(\frac{\gamma}{\rho}\frac{1}{c_1}p + (\frac{\gamma}{\rho}\frac{1}{k_1} + \frac{1}{c_1})\dot{p} + \frac{1}{k_1}\ddot{p} - \mathbf{n} \cdot \dot{\mathbf{v}}^m)dS = 0 \tag{23}$$

where the inverse of $\frac{1}{k_1}$ and $\frac{1}{c_1}$ are the spring and dashpot parameters of the acoustic material layer respectively, vector $\mathbf{n}$ represents the *inward* normal to the acoustic medium at the boundary and $\dot{\mathbf{v}}^m$ denotes the acceleration of the structure adjoining with acoustic medium.

The level of the absorption at boundary is controlled by the acoustic impedance of acoustic material. Such conditions are applied on S_{fr} surface in the following form (in absence of internal friction $\gamma = 0$)

$$-\mathbf{n} \cdot \mathbf{v} = (\frac{1}{k_1}\dot{p} + \frac{1}{c_1}p), \tag{24}$$

[4] for detailed information see [6]

thus the proper choose of k_1 and c_1 coefficients allows us to obtain the demanded absorption.

The results shown above were obtained under the assumption that the room temperature is considered, so the following values describing the properties of acoustic medium were established: the density of air $\rho = 1.20\frac{kg}{m^3}$, and the bulk modulus $K = 141.8\frac{kN}{m^2}$.

In Fig. (4) the results from FE analyses are presented. The *incident wave* presents the acoustic pressure distribution at time instant when the planar wave had not reach the boundary yet. The *reflected wave* presents the the acoustic pressure field after reflection (only reflected wave is presented) and the influence of corrugation is observed. In the last part of Fig. (4) the resultant acoustic pressure distribution is presented to proof that such field of pressure is possible to obtain by homogenized boundaries (the wave front is almost parallel to boundary). Similarly to the analytical solution the amplitude of reflected wave was from the range $0.96 \div 0.98$.

The substitute impedance conditions for numerical analyses are obtained from Eqn. (24). If we neglect the out of phase term we will obtain

$$-\mathbf{n} \cdot \mathbf{v} = \frac{1}{c_1} p, \tag{25}$$

or using above introduced notation

$$-\mathbf{n} \cdot \mathbf{v} = \frac{1}{Z_2} p. \tag{26}$$

As before if we replace in Egn. (26) Z_2 by Z_2^* from Eqn. (20) and in numerical model we substitute the corrugated boundary by planar one the process of boundary homogenization will be completed.

3 Conclusions

The analytical solution and numerical computations point out that the homogenized boundaries in problems where the acoustic planar wave traveling is considered can give reasonable results. Beyond any doubt is that the corrugation of boundary (even very "flat") can have considerable influence on acoustic pressure distribution. Figure (5) indicates the case where the acoustic wavelength and the geometry of corrugation have such dimensions that homogenization would not have sense (the higher order terms of Fourier series (see Egn. (9)) are of great importance). Thus the homogenization can be applied only for special cases of boundaries and

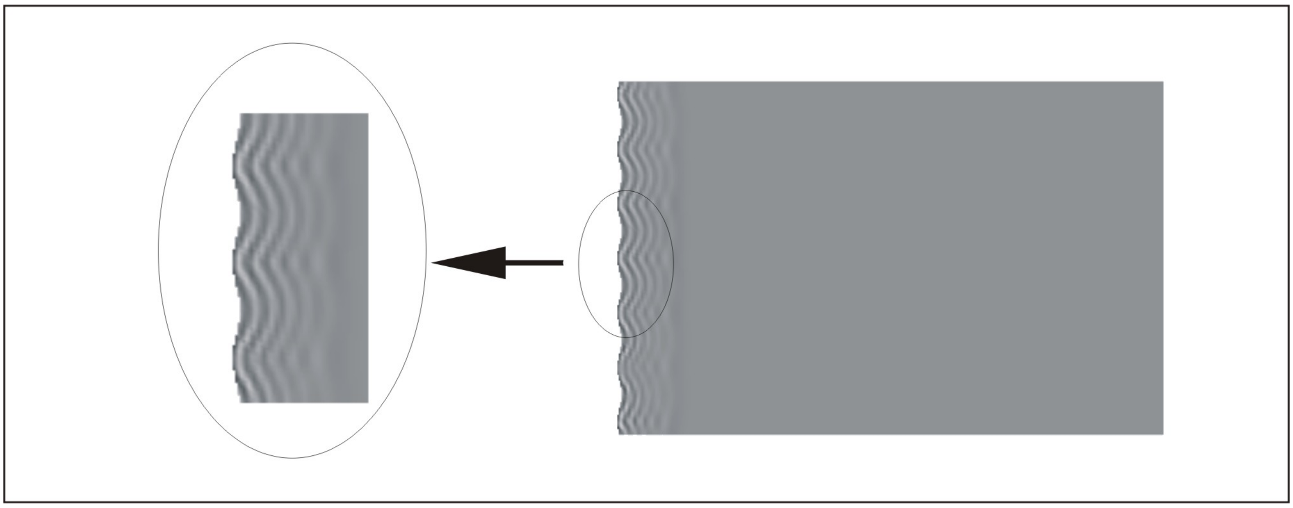

Figure 5: The influence of higher order waves

under the assumption that waves characteristics (i.e. wavelength) are properly fitted to corrugation dimensions.

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